

Evaluation of Approximation of Fisher Information Matrix in Poisson Mixture Model using High Performance Computing

Qing Ji[†], Matthias K. Gobbert[†], Andrew M. Raim^{*}, Nagaraj K. Neerchal[†], Jorge G. Morel[†]

[†] Department of Mathematics and Statistics, University of Maryland, Baltimore County

^{*} United States Census Bureau

PROJECT SUMMARY

- Fisher information matrix(FIM) is an essential part in the computation of maximum likelihood estimation(MLE) as well as in obtaining their standard errors
- Computation of FIM is resource intensive in some widely used models such as mixture distributions
- Neerchal and Morel(1993), Raim (2014), and Raim et. al.(2014) have provided approximations to the FIM
- In this project, we apply this approximation idea to a mixture of two Poisson distributions
- A program in C with MPI is designed to test the performance of our approximation under various selections of parameter values

BACKGROUND

$$X_1, \dots, X_n \stackrel{iid}{\sim} f(x | \theta),$$

$$\theta = (\theta_1, \dots, \theta_k) \in \Theta, \quad x \in \mathcal{X}$$

- The likelihood function is

$$\mathcal{L}(\theta | \mathbf{x}) = \prod_{i=1}^n f(x_i | \theta),$$

where $\mathbf{x} = (x_1, \dots, x_n)$ is observed data

- The score function is given by

$$S(\theta) = E \left\{ \frac{\partial}{\partial \theta} \log \mathcal{L}(\theta | \mathbf{x}) \right\}$$

- The Fisher Information matrix(FIM) is:

$$I(\theta) = -E \left[\frac{\partial^2}{\partial \theta \partial \theta'} \log \mathcal{L}(\theta | \mathbf{x}) \right]$$

$$= \text{var} \left\{ \frac{\partial}{\partial \theta} \log \mathcal{L}(\theta | \mathbf{x}) \right\}$$

- The Fisher scoring algorithm of obtaining MLE is given by:

$$\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} - [I(\hat{\theta}^{(i)})]^{-1} S(\hat{\theta}^{(i)})$$

POISSON MIXTURE MODEL

- The probability function of the mixture of two Poisson distributions is given by:

$$p(x; \theta) = \pi \frac{\lambda_1^x e^{-\lambda_1}}{x!} + (1 - \pi) \frac{\lambda_2^x e^{-\lambda_2}}{x!}$$

- The log-likelihood function for $\mathbf{x} = [x_1 x_2 \dots x_n]'$ is given by:

$$\log \mathcal{L}(\theta | \mathbf{x}) = \sum_{i=1}^n \log \left\{ \pi \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!} + (1 - \pi) \frac{\lambda_2^{x_i} e^{-\lambda_2}}{x_i!} \right\},$$

- The score function is given by:

$$S(\theta; \mathbf{x}) = \frac{\partial}{\partial \theta} \log \mathcal{L}(\theta | \mathbf{x})$$

$$= \frac{\partial}{\partial \theta} \sum_{i=1}^n \log \left\{ \pi \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!} + (1 - \pi) \frac{\lambda_2^{x_i} e^{-\lambda_2}}{x_i!} \right\}$$

- The true FIM for $n = 1$:

$$-E \left\{ \frac{\partial^2}{\partial \theta \partial \theta'} \log \mathcal{L}(\theta | \mathbf{x}) \right\}$$

$$= \text{var} \left\{ S(\theta; \mathbf{x}) \right\}$$

$$= - \sum_{x=0}^{\infty} \{ S(\theta; \mathbf{x}) S'(\theta; \mathbf{x}) \} p(x; \theta)$$

WHY CONSIDER THIS MODEL?

- Mixture models are very useful in modeling populations consisting of two or more distinct sub-groups but the group labels are not available
- Mixture models are also used, as shown in Raim(2014), to analyze data from multiple studies with a common objective but possibly different target population (Meta Analysis)
- Mixture models are also useful in modeling count data exhibiting more variance than accommodated by the binomial or Poisson models, a phenomenon known as overdispersion

IMPLEMENTATION DETAILS

- Let's consider several combinations of values for the parameters:

$$\lambda_1, \lambda_2 \in \{0, 1, 2, \dots, 500\}$$

$$\pi \in \{0.1, 0.2, \dots, 0.5\}$$

- Compute the true FIM, I , by truncation method:

$$\sum_{x=0}^N \left[\frac{\partial}{\partial \theta_i} \ln p(x; \theta) \frac{\partial}{\partial \theta_j} \ln p(x; \theta) \right] p(x; \theta),$$

$$\text{with } p(x; \theta) = \pi \frac{\lambda_1^x e^{-\lambda_1}}{x!} + (1 - \pi) \frac{\lambda_2^x e^{-\lambda_2}}{x!}$$

where N is a very large number chosen for truncation

- The approximation is given by:

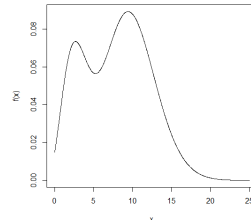
$$I_A = \begin{bmatrix} \frac{n\pi_1}{\lambda_1} & 0 & 0 \\ 0 & \frac{n\pi_2}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\pi(1-\pi)} \end{bmatrix}$$

- Quality of the approximation is measured using Frobenius norm:

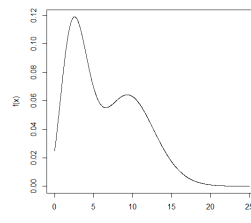
$$\|I_A - I\|_F = \sqrt{\sum_{a=1}^3 \sum_{b=1}^3 (I_A[a, b] - I[a, b])^2}$$

where $I_A[a, b]$ and $I[a, b]$ denote the $(a, b)^{th}$ entry in the approximate and exact FIM

EXAMPLES OF POISSON MIXTURE



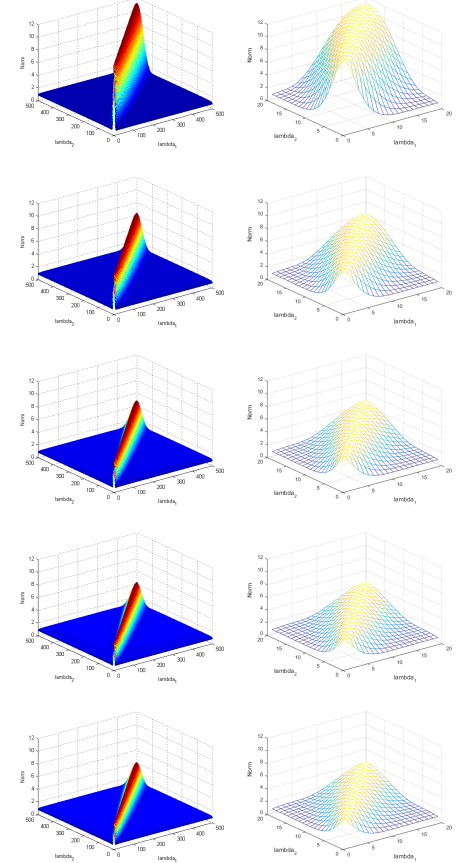
The density of a mixture Poisson for $\lambda_1 = 3, \lambda_2 = 10$ and $\pi = 0.3$



The density of a mixture Poisson for $\lambda_1 = 3, \lambda_2 = 10$ and $\pi = 0.5$

RESULTS

Below are two sets of 3D plots addressing the relationship between $\|I_A - I\|_F$ and (λ_1, λ_2) as π increased from 0.1 to 0.5 (top to bottom). The complete picture of this simulation is displayed, $\lambda \in [0, 500]$ on the left and $\lambda \in [0, 20]$ and the domain of λ on the right



- The $\|I_A - I\|_F$ remains very low and almost fixed when λ_1 and λ_2 are far apart
- $\|I_A - I\|_F$ increases dramatically once two λ 's are within certain proximity
- $\|I_A - I\|_F$ reaches maximum when $\lambda_1 = \lambda_2$
- This pattern is not affected by the individual λ values but rather the closeness of the two λ 's
- The value of π only decreases the maximum value as $\pi \rightarrow 0.5$, the pattern remains the same

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