Evaluation of Approximation of Fisher Information Matrix in Poisson Mixture Model using High Performance Computing

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Project Summary

- Fisher information matrix(FIM) is an essential part in the computation of maximum likelihood estimation(MLE) as well as in obtaining their standard errors
- Computation of FIM is resource intensive in some widely used models such as mixture distributions
- Neerchal and Morel(1993), Raim (2014), and Raim et. al.(2014) have provided approximations to the FIM
- In this project, we apply this approximation idea to a mixture of two Poisson distributions
- A program in C with MPI is designed to test the performance of our approximation under various selections of parameter values

Background

$$X_1, \dots, X_n \overset{iid}{\sim} f(x \mid \boldsymbol{\theta}),$$
$$\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \in \Theta, \quad x \in \mathcal{X}$$

- The likelihood function is

$$\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{x}) = \prod_{i=1}^n f(x_i \mid \boldsymbol{\theta}),$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$ is observed data
- The score function is given by

$$S(\boldsymbol{\theta}) = E\Big\{\frac{\partial}{\partial \boldsymbol{\theta}} log\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{x})\Big\}$$

- The Fisher Information matrix(FIM) is:

$$\boldsymbol{I}(\boldsymbol{\theta}) = -\,E\Big[\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'} log\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{x})\Big]$$
$$= \mathbf{var}\Big\{\frac{\partial}{\partial \boldsymbol{\theta}} log\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{x})\Big\}$$

- The Fisher scoring algorithm of obtaining MLE is given by:

$$\hat{\boldsymbol{\theta}}^{(i+1)} = \hat{\boldsymbol{\theta}}^{(i)} - [\boldsymbol{I}(\hat{\boldsymbol{\theta}}^{(i)})]^{-1} S(\hat{\boldsymbol{\theta}}^{(i)})$$

Poisson Mixture Model

- The probability function of the mixture of two Poisson distributions is given by:

$$p(x;\boldsymbol{\theta}) = \pi\frac{\lambda_1^x e^{-\lambda_1}}{x!} + (1-\pi)\frac{\lambda_2^x e^{-\lambda_2}}{x!}$$

- The log-likelihood function for $\boldsymbol{x} = [x_1 x_2 \dots x_n]'$ is given by:

$$log\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{x}) = \sum_{i=1}^n log\{\pi\frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!} + (1-\pi)\frac{\lambda_2^{x_i} e^{-\lambda_2}}{x_i!}\},$$

- The score function is given by:

$$S(\boldsymbol{\theta}; \boldsymbol{x}) = \frac{\partial}{\partial\boldsymbol{\theta}} log\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{x})$$
$$= \frac{\partial}{\partial\boldsymbol{\theta}} \sum_{i=1}^n log\{\pi\frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!} + (1-\pi)\frac{\lambda_2^{x_i} e^{-\lambda_2}}{x_i!}\}$$

- The true FIM for $n = 1$:

$$-\,E\Big\{\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'} log\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{x})\Big\}$$
$$= \mathbf{var}\Big\{S(\boldsymbol{\theta}; x)\Big\}$$
$$= -\sum_{x=0}^{\infty}\{S(\boldsymbol{\theta}; x) S'(\boldsymbol{\theta}; x)\} p(x;\boldsymbol{\theta})$$

Why Consider This Model?

- Mixture models are very useful in modeling populations consisting of two or more distinct subgroups but the group labels are not available
- Mixture models are also used, as shown in Raim(2014), to analyze data from multiple studies with a common objective but possibly different target population (Meta Analysis)
- Mixture models are also useful in modeling count data exhibiting more variance than accommodated by the binomial or Poisson models, a phenomenon known as overdispersion

Implementation Details

- Let's consider several combinations of values for the parameters:

$$\lambda_1, \lambda_2 \in \{0, 1, 2, \dots, 500\}$$
$$\pi \in \{0.1, 0.2, \dots, 0.5\}$$

- Compute the true FIM, $\boldsymbol{I}$, by truncation method:

$$\sum_{x=0}^{N}\Big[\frac{\partial}{\partial\theta_i}\ln p(x;\boldsymbol{\theta})\frac{\partial}{\partial\theta_j}\ln p(x;\boldsymbol{\theta})\Big]p(x;\boldsymbol{\theta}),$$

with $p(x;\boldsymbol{\theta}) = \pi\frac{\lambda_1^x e^{-\lambda_1}}{x!} + (1-\pi)\frac{\lambda_2^x e^{-\lambda_2}}{x!}$

where N is a very large number chosen for truncation
- The approximation is given by:

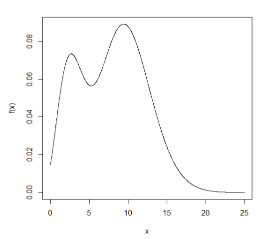
$$\mathbf{I_A} = \begin{bmatrix} \frac{n\pi_1}{\lambda_1} & 0 & 0 \\ 0 & \frac{n\pi_2}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\pi(1-\pi)} \end{bmatrix}$$

- Quality of the approximation is measured using Frobenius norm:

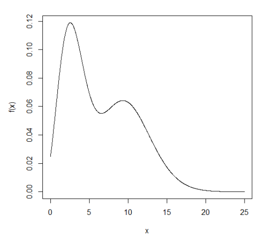
$$\|I_A - I\|_F = \sqrt{\sum_{a=1}^3 \sum_{b=1}^3 (I_A[a,b] - I[a,b])^2}$$

where $I_A[a,b]$ and $I[a,b]$ denote the $(a,b)^{th}$ entry in the approximate and exact FIM

Examples of Poisson Mixture



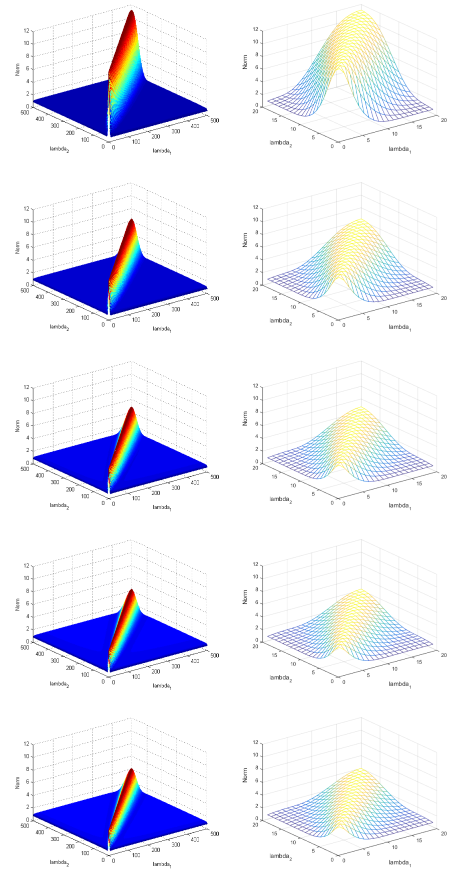
The density of a mixture Poisson for $\lambda_1 = 3, \lambda_2 = 10$ and $\pi = 0.3$



The density of a mixture Poisson for $\lambda_1 = 3, \lambda_2 = 10$ and $\pi = 0.5$

Results

Below are two sets of 3D plots addressing the relationship between $\|I_A - I\|_F$ and (λ_1, λ_2) as π increased from 0.1 to 0.5 (top to bottom). The complete picture of this simulation is displayed, $\lambda \in [0, 500]$ on the left and $\lambda \in [0, 20]$ and the domain of $lambda$ is on the right



- The $\|I_A - I\|_F$ remains very low and almost fixed when λ_1 and λ_2 are far apart
- $\|I_A - I\|_F$ increases dramatically once two λ's are within certain proximity
- $\|I_A - I\|_F$ reaches maximum when $\lambda_1 = \lambda_2$
- This pattern is not affected by the individual λ values but rather the closeness of the two λ's
- The value of π only decreases the maximum value as $\pi \to 0.5$, the pattern remains the same

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Reference

- Raim, A.M., Neerchal, N.K., and Morel, G.J., An approximation to the information matrix of exponential family finite mixtures, *Statistical Methodology*, 18:115-130, 2014.
- Raim, A.M., Computational Methods in Finite Mixtures Using Approximate Information and Regression Linked to the Mixture Mean, Unpublished PhD. Thesis, University of Maryland Baltimore County, 2014.
- Morel, J.G., Neerchal, N.K., A finite mixture distribution for modeling multinational extra variation *Biometrika*, 80(2):363-371, 1993.